J. Fluid Mech. (1998), *vol.* 360, *pp.* 229–248. Printed in the United Kingdom

Turbulent gravitational convection from a point source in a non-uniformly stratified environment

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(Received 1 August 1997 and in revised form 4 December 1997)

We examine the turbulent gravitational convection which develops above a point source of buoyant fluid in a stably stratified environment in which the buoyancy frequency varies with height according to $N^2 = N_s^2(z/z_s)^\beta$. This generalizes the classical model of turbulent buoyant plumes rising through uniform and uniformly stratified environments originally developed by Morton *et al.* (1956). By analogy, the height of rise of a plume with initial buoyancy flux F_s has the form $H_p = A_p\epsilon_p^{-1/2}F_s^{1/4}N_s^{-3/4}h_p(\lambda, \beta)$ where ϵ_p is the entrainment constant for plume motion, A_p is an $O(1)$ constant, and the non-dimensional plume height, h_p is a function of $\lambda = A_p\epsilon_p^{-1/2}F_s^{1/4}N_s^{-3/4}/z_s$ and β.

In the case $\beta > 0$, the stratification becomes progressively stronger with height, and so plumes are always confined within a finite distance above the origin. Furthermore, the non-dimensional height of rise h decreases with λ. In contrast, in the case $\beta < 0$, the stratification becomes progressively weaker with height, and so the non-dimensional plume height increases monotonically with λ. For slowly decaying stratification, $\beta > -8/3$, the motion is confined within a finite distance above the source. However, for each value of β with $\beta < -8/3$, there is a critical value $\lambda_c(\beta)$ such that for $\lambda < \lambda_c$ a plume is confined to a region near the source while for $\lambda \geqslant \lambda_c$ the motion is unbounded. In the unbounded case, the motion asymptotes to the solution for a buoyant plume rising through a uniform environment, with asymptotic buoyancy flux $F_\infty(\lambda) < F_s$. We show that in the limiting case $\lambda = \lambda_c$, dividing bounded and unbounded motion, as $z \to \infty$ the plume asymptotes to a new similarity solution of the second kind which describes the motion of a plume in a non-uniformly stratified environment. These similarity solutions are unstable in the sense that small perturbations to the initial conditions result in very different behaviour far from the source.

Analogous results for an instantaneous release of buoyant fluid from a point source, which forms a thermal, are also presented. The model is applied to describe the motion of plumes and thermals in the upper ocean and in naturally ventilated buildings since in both cases the stratification is typically non-uniform.

1. Introduction

The behaviour of continuous and instantaneous releases of buoyant fluid, which are commonly referred to as plumes and thermals, rising from a point source of buoyancy is well understood. For turbulently convecting fluid far from an isolated (continuous) source of buoyancy in an unstratified medium, Zeldovich (1937) demonstrated the existence of asymptotic, self-similar solutions for the velocity and density within the isolated 'plume'. The classical models of Morton, Taylor & Turner (1956) further describe the motion of both plumes and thermals in unstratified and uniformly stratified environments, and have been successfully tested with controlled laboratory experiments and naturally occurring examples over a wide range of scales (Briggs 1969; Turner 1986; Woods 1995). However, there are a number of interesting situations in which buoyant fluid may be released into an environment in which the density stratification is non-uniform. For example, the combination of heat sources and air vents in buildings often leads to non-uniform thermal stratification through filling-box-type processes (Baines & Turner 1969; Linden & Cooper 1996). Another example concerns ocean–atmosphere interactions which often lead to elevated water temperatures above the thermocline, and hence much larger stratification than in deeper waters (Emery, Lee & Magaard 1984; Morison *et al.* 1992). The elevated stratification near the surface may arrest the descent of sewage discharges, dense turbidity currents and the dense saline plumes produced during sea-ice formation (Morison *et al.* 1992). The effects of non-uniform stratification may also be of importance in the lower atmosphere in still conditions, where inversions, or regions of high stratification develop just above the ground (Gill 1982).

In order to examine the effects of non-uniform stratification, in §2 we generalize the model first developed by Morton *et al.* (1956) to describe the ascent of a turbulent buoyant plume in an environment in which the buoyancy frequency varies with height, $N(z)$. Morton *et al.* (1956) identified that the motion of a turbulent plume in an unstratified environment is a self-similar function of height z, and so it is natural to examine the special class of ambient stable stratifications described by

$$N^2(z) = N_s^2(z/z_s)^\beta, \tag{1.1}$$

where $N_s^2 > 0$. If we consider a plume whose source is located at $z = z_s$ then N_s represents the buoyancy frequency at the source, and z_s represents the characteristic length scale of variability of the stratification.

In §3, we use the generalized model to illustrate how the behaviour of a turbulent buoyant plume, of given initial buoyancy flux, varies with z_s and β, identifying conditions under which the motion is bounded and unbounded. We then present a new family of similarity solutions associated with the transition between bounded and unbounded motion. In §5, we present some analogous results for discrete thermal clouds. In §6, we apply the results of the model to several situations of practical interest and we draw some conclusions in §7. For completeness, in an Appendix we illustrate analogous results for turbulent plumes and thermals rising through an environment in which the stratification decays exponentially with height.

2. A model of a turbulent buoyant plume in a non-uniform environment

Following Morton *et al.* (1956), we consider an environment in which the variations in density are small compared to the background density, so that we may adopt the Boussinesq approximation. In this limit, the mass flux, $\pi\rho_s Q$, momentum flux, $\pi\rho_s M$, and buoyancy flux, $\pi\rho_s F$, may be defined by

$$Q = 2\int_0^\infty wr\,\mathrm{d}r; \tag{2.1}$$

$$M = 2\int_0^\infty w^2 r\,\mathrm{d}r; \tag{2.2}$$

$$F = \frac{2g}{\rho_s}\int_0^\infty u\Delta\rho r \mathrm{d}r, \tag{2.3}$$

where w is the vertical velocity within the plume, $\Delta\rho$ the density deficit of the plume relative to the environment, ρ_s is a reference density, and the integral is over the whole cross-sectional area of the plume. We adopt the entrainment assumption that the horizontal inflow velocity u_e is linearly related to the vertical velocity w within the plume:

$$u_e = \epsilon_p w. \tag{2.4}$$

The entrainment constant ϵ_p has been determined experimentally to have value ~ 0.09 for both uniform and uniformly stratified environments (Morton *et al.* 1956; Turner 1986). The equations for the conservation of mass, momentum and buoyancy averaged over a horizontal cross-section may then be expressed in the form

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = 2\epsilon_p M^{1/2}, \tag{2.5}$$

$$\frac{\mathrm{d}M}{\mathrm{d}z} = \frac{FQ}{M}, \tag{2.6}$$

$$\frac{\mathrm{d}F}{\mathrm{d}z} = Q\frac{g}{\rho_s}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}, \tag{2.7}$$

where $\bar{\rho}$ represents the density of the environment and ρ_s is a fixed reference density. Adopting the relation (1.1) for the buoyancy frequency N of the environment, and remembering that the buoyancy frequency is defined as

$$N^2(z) = -\frac{g}{\rho_s}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}, \tag{2.8}$$

(2.7) may be re-written as

$$\frac{\mathrm{d}F}{\mathrm{d}z} = -QN_s^2(z/z_s)^\beta. \tag{2.9}$$

Morton *et al.* (1956) considered the case in which the buoyancy frequency is a constant, $\beta = 0$, in which case a pure point source plume, with zero initial volume flux and specific momentum flux, ascends a finite height H_{mp} given in terms of the source specific buoyancy flux F_s by

$$H_{mp} = 2.57H_p, \tag{2.10}$$

where

$$H_p \equiv (2\epsilon_p)^{-1/2}{F_s}^{1/4}N_s^{-3/4}. \tag{2.11}$$

H_p is a characteristic length scale of the plume height of rise which may be used to define non-dimensional variables

$$\hat{F} = \frac{F}{F_s}, \quad \hat{Q} = \frac{Q}{(2\epsilon_p)^{4/3}F_s^{1/3}H_p^{5/3}}, \quad \hat{M} = \frac{M}{(2\epsilon_p)^{2/3}F_s^{2/3}H_p^{4/3}}, \tag{2.12a–c}$$

with

$$\hat{z} = \frac{z}{H_p} \quad \text{and} \quad \hat{N} = \frac{N}{N_s}. \tag{2.13a,b}$$

These lead to the non-dimensional governing equations (cf. (2.5)–(2.7))

$$\frac{\mathrm{d}}{\mathrm{d}\hat{z}}\hat{Q} = \hat{M}^{1/2}, \quad \hat{M}\frac{\mathrm{d}}{\mathrm{d}\hat{z}}\hat{M} = \hat{F}\hat{Q}, \quad \frac{\mathrm{d}}{\mathrm{d}\hat{z}}\hat{F} = -\hat{Q}\hat{N}^2, \quad \hat{N}^2 = (\lambda\hat{z})^\beta. \tag{2.14a–d}$$

The parameter λ represents the ratio of the scale height of rise of a plume with initial specific buoyancy flux F_s in a uniformly stratified environment with buoyancy frequency N_s, to z_s, the scale height of the environmental stratification:

$$\lambda \equiv \frac{H_p}{z_s}. \tag{2.15}$$

In the remainder of this paper, we work with these non-dimensional variables, and so, for convenience, we now drop the hat notation to denote a non-dimensional quantity; all quantities are non-dimensional unless explicitly stated otherwise.

3. Sensitivity of the plume rise to ambient stratification

Equation (2.14*c*) implies that, in a stably stratified environment, whatever the initial volume and specific momentum fluxes, the specific buoyancy flux decreases with height owing to the entrainment of relatively dense ambient fluid. If the specific buoyancy flux decreases to zero and becomes negative, then the plume specific momentum flux decreases and the motion eventually comes to rest at a finite height; otherwise, the plume continues rising.

In order to compare the predictions of the model directly with the calculations of Morton *et al.* (1956) for a point-source plume rising in a uniformly stratified environment, we choose initial conditions corresponding to a point-source plume,

$$Q = 0, \quad M = 0, \quad F = 1 \quad \text{at} \quad z = 1/\lambda\,. \tag{3.1}$$

The non-dimensional height of rise,

$$h_p(\lambda, \beta) = z_p - 1/\lambda, \tag{3.2}$$

is defined as the height at which the specific momentum flux $M(z_p) = 0$. The initial position, $z = 1/\lambda$, has been chosen so that both F and $N = 1$ (see (2.12) and (2.13)). In the special case $\beta = 0$, corresponding to a uniformly stratified environment,

$$h_p \equiv h_{mp} = 2.57. \tag{3.3}$$

In figure 1, we illustrate how the height of rise of a plume varies with β for several values of λ, comparing the actual height of rise to this classical result (see Morton *et al.* 1956).

3.1. *Increasing stratification:* $\beta > 0$

In the case $\beta > 0$, the ambient stratification increases with height and so the plume height is smaller than for a uniformly stratified environment, $h_p < h_{mp}$. For small values of $\lambda(\ll 1)$, the scale over which the stratification increases, z_s, is appreciably greater than the height of rise of the plume through a uniformly stratified environment with buoyancy frequency equal to that at the source. Therefore, the plume does not rise sufficiently far to experience the increasing stratification, and the decrease of the plume height is relatively small, $h_p \sim h_{mp}$. However, for larger values of λ ($\gg 1$), the environmental stratification increases much more rapidly, and so the plume height becomes considerably smaller.

3.2. *Decreasing stratification:* $\beta < 0$

For $\beta < 0$, the ambient stratification becomes progressively weaker with height and so plumes tend to rise higher than in a uniformly stratified environment, $h_p > h_{mp}$. For $\lambda \gg 1$, the stratification decreases rapidly over the height to which the plume

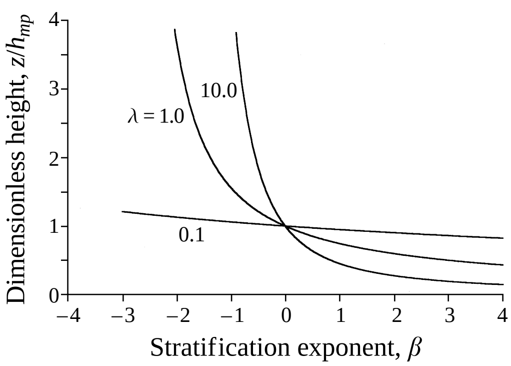

FIGURE 1. Non-dimensional height of rise of turbulent buoyant plume in an environment of non-uniform stratification (defined by (1.1)) as a function of β. Curves are given for three values of λ (defined by equation (2.15)), namely 0.1, 1.0 and 10.0. For reference, the plume heights have been scaled relative to the height of rise H_{mp} (defined by (2.10)) of a plume rising in a uniformly stratified environment (cf. Zeldovich 1937; Morton *et al.* 1956).

would rise if the environment were of uniform stratification. As a result, the plume rises much higher, $h_p \gg h_{mp}$. In contrast, for $\lambda \ll 1$, the plume does not experience a significant change in the stratification before it has reached its maximum height of rise, and so $h_p \sim h_{mp}$.

3.3. *Bounded and unbounded motion for* $\beta < -8.3$

Further numerical calculations have identified that for each value of $\beta < -8/3$, there is a critical value $\lambda = \lambda_c(\beta)$, such that for $\lambda > \lambda_c(\beta)$, the plume is unbounded even though it is rising through a stratified environment. This is because the stratification decreases to very small values some distance above the source so that the ambient density becomes nearly uniform and the material can therefore remain buoyant even though it is entraining relatively dense fluid near the source. In fact, the motion asymptotes to that of a buoyant plume in an unstratified environment. However, owing to the stratification, the asymptotic specific buoyancy flux, $F_\infty(\lambda)$, is smaller than that at the source. This implies that as $z \to \infty$, $F_\infty(\lambda) < 1$ while the volume flux and specific momentum fluxes have the asymptotic form (cf. Morton *et al.* 1956)

$$Q_\infty \sim \frac{3}{5}\left(\frac{9}{20}\right)^{1/3} F_\infty^{1/3}(\lambda) z^{5/3}, \tag{3.4}$$

$$M_\infty \sim \left(\frac{9}{20}\right)^{2/3} F_\infty^{2/3}(\lambda) z^{4/3}, \tag{3.5}$$

for $z \gg 1$. Numerical solution of the governing equations (2.14*a*–*c*) shows that $F_\infty(\lambda) \to 1$ as $\lambda \to \infty$ and that $F_\infty(\lambda) \to 0$ as $\lambda \to \lambda_c(\beta)$ from above, as shown in figure 2(*a*).

The calculations have also identified that for $\lambda < \lambda_c(\beta)$, the plume motion is bounded (figure 2*b*), while the total height of rise increases as $\lambda \to \lambda_c(\beta)$ from below. This is because the stratification decays so slowly with height that the plume specific buoyancy flux falls to zero before the stratification has decayed sufficiently.

In figure 3, we show how the critical value $\lambda_c(\beta)$ at which plumes are just able to overcome the stratification and become unbounded varies with β. It is seen that as $|\beta|$ increases, and the stratification decays more rapidly with height, the critical value λ_c becomes smaller.

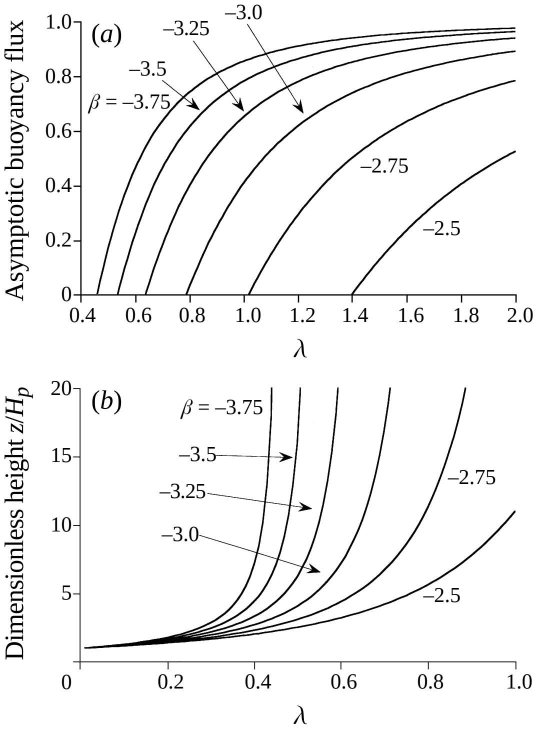

FIGURE 2. (a) The asymptotic non-dimensional specific buoyancy flux, $F_\infty < 1$, as a function of λ, for $\lambda > \lambda_c$. (b) Variation of the height of rise of a buoyant plume with $\lambda < \lambda_c$ as a function of λ. Curves are given for $\beta = -2.5, -2.75, -3.0, -3.25, -3.5$ and -3.75.

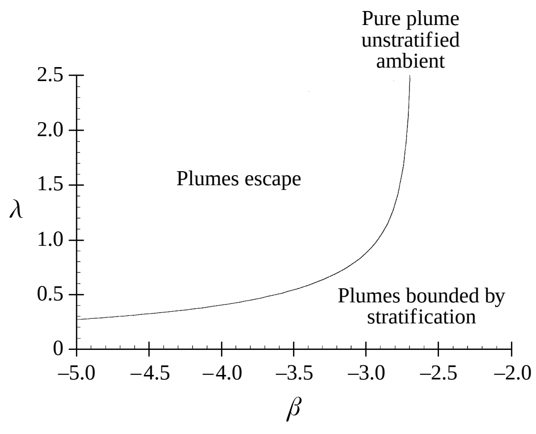

FIGURE 3. Variation of the critical value λ_c separating bounded and unbounded solutions as a function of β.

Essentially, as the rate at which stratification decays with height increases, the maximum scale height of the stratification, z_s, for which plumes of a given specific buoyancy flux are unbounded, also increases.

The qualitative difference between bounded and unbounded plumes, which arises for $\beta < -8/3$, is of particular interest. In the next section, we show that, for $-4 < \beta < -8/3$, there is a new family of similarity solutions of (2.14) in which the specific buoyancy flux decays monotonically to zero as $z \to \infty$. In the special limit $\lambda = \lambda_c$, the numerical solutions of this section asymptote towards these similarity solutions.

4. Self-similar plumes in non-uniform environments

Direct calculation shows that when $-4 < \beta < -8/3$, there is a family of similarity solutions, of the second kind, for (2.14) of the form

$$Q_u(z) = Q_0 z^q, \quad M_u(z) = M_0 z^m, \quad F_u(z) = F_0 z^f, \tag{4.1a–c}$$

where

$$Q_0 = \frac{\lambda^{\beta/2}}{(|f|mq^4)^{1/2}}, \quad M_0 = \frac{\lambda^{\beta}}{(|f|mq^2)}, \quad F_0 = \frac{\lambda^{3\beta/2}}{(|f|^3mq^4)^{1/2}}, \tag{4.2a–c}$$

and

$$q = \frac{(\beta+6)}{2}, \quad m = \beta + 4, \quad f = \frac{(3\beta+8)}{2}. \tag{4.3a–c}$$

Before describing these solutions in more detail, it is interesting to note that Batchelor (1954) considered the related, but distinct problem of motion through a statically unstable ambient, $N_s^2 < 0$. He presented a somewhat similar family of similarity solutions, but valid for $\beta > -8/3$, and which had zero buoyancy flux at $z = 0$. He interpreted his solutions as a mathematical demonstration of the fact that 'the development of a turbulent heat plume in an unstable atmosphere does not need a heat source'. We, conversely, are interested in statically stable environments, for which these similarity solutions apply when $-4 < \beta < -8/3$, and which have finite buoyancy frequency and specific buoyancy flux at the source, $z = 1/\lambda$.

Note that except for the limits $\beta = -4$ and $-8/3$, m and q are positive, while f is negative, so that F_u decreases while both M_u and Q_u increase with z. Such plumes are able to propagate indefinitely because the stratification decays sufficiently rapidly that the net specific buoyancy flux always remains positive.

4.1. *Limit of an unstratified environment, $\beta = -8/3$*

In the limit $\beta = -8/3$, $f \to 0$, and so for finite λ, F_0 and $M_0 \to \infty$. However, in the special case $\lambda f^{3/8} = O(1)$ as $f \to 0$, the solution coincides with that for a plume rising in an unstratified environment with finite constant specific buoyancy flux F_0 (Zeldovich 1937; Morton *et al.* 1956)

$$Q_0 = \frac{3}{5}\left(\frac{9F_0}{20}\right)^{1/3} z^{5/3}, \quad M_0 = \left(\frac{9F_0}{20}\right)^{2/3} z^{4/3}. \tag{4.4a,b}$$

This has the same form as the asymptotic solution defined by equations (3.4) and (3.5) discussed in the previous section.

4.2. *Limit of a momentum jet, $\beta = -4$*

The limit $\beta = -4$ is also singular for finite λ. However, in the special case $\lambda m^{1/4} = O(1)$ as $\lambda \to 0$, then the similarity solution corresponds to a turbulent momentum jet rising through an unstratified environment and the solution may be expressed in the simple form

$$Q_0 = z, \quad M_0 = 1, \quad F_0 = 0. \tag{4.5a–c}$$

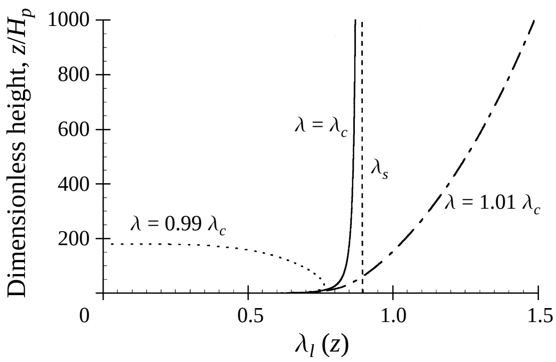

FIGURE 4. Variation of the local value of $\lambda_l(z)$ (defined by (4.16)) calculated in the numerical solutions for a plume rising in a stably stratified ambient fluid with $\beta = -3.5$ for three different values of λ. For this value of β, λ_c is approximately 0.543, and $\lambda_s = 0.897$. For $\lambda < \lambda_c$, the plume is bounded and so λ_l passes through zero. For $\lambda > \lambda_c$ the motion is unbounded, and $\lambda_l \to \infty$, while in the critical case $\lambda = \lambda_c$ the $\lambda(z) \to \lambda_s$ (shown by a vertical dashed line) and the plume converges towards the similarity solution, defined by (4.1).

4.3. *Comparison of the similarity solutions with the numerical solutions for* $-4 < \beta < -8/3$

The numerical solutions of §3, with $\lambda = \lambda_c$, represent the distinct limit in which the motion of the plume is unbounded, but for which the specific buoyancy flux decays towards zero as $z \to \infty$ (figure 2). We might therefore expect that these solutions coincide with the similarity solutions (4.1) as $z \to \infty$. In order to compare the numerical solutions with the similarity solutions,and in particular to determine whether they converge towards the similarity solution with height, it is convenient to define the quantity

$$\lambda_l(z) = \frac{F^{1/4}(z)N^{-3/4}(z)}{z} = F^{1/4}(z)\lambda^{-3\beta/8}z^{-1-3\beta/8}. \tag{4.6}$$

For general λ, in the numerical solutions of §3, $\lambda_l(z)$ evolves with height in the plume, but, for the self-similar solutions, λ_l has the constant value

$$\lambda_s = (q^4 m|f|^3)^{-1/8} \tag{4.7}$$

at all heights.

For the numerical calculations of §3, we find three types of behaviour of $\lambda_l(z)$, depending on the initial value of λ (figure 4). When $\lambda < \lambda_c$ at the source, we find that as the motion becomes suppressed by the stratification, λ_l decreases to zero at the height at which the plume becomes dense. Conversely, if $\lambda > \lambda_c$ at the source, then as $z \to \infty$, λ_l increases indefinitely with height. For the critical condition $\lambda = \lambda_c$ at the source, the numerical calculations suggest that as $z \to \infty$, $\lambda_l \to \lambda_s$ again and so these particular plume solutions do converge to the similarity solutions given by (4.1), once the information about the initial conditions is lost through entrainment, as expected for

$$z \gg (\lambda_c^{3\beta/2}\lambda_s^4)^{1/|f|}. \tag{4.8}$$

In figure 5, we show how M_0 (solid line) varies with β, in the range $-4 < \beta < -8/3$ for these critical solutions with $\lambda = \lambda_c$. The figure also shows (with a dashed line) the variation of the asymptotic specific momentum flux M_∞ (as defined by (4.22) in §4.6) with β for flows (with $\beta < -4$) where the similarity solutions defined by (4.1) cannot

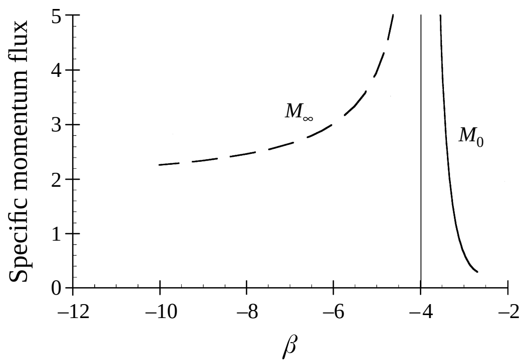

FIGURE 5. Variation of M_0 (defined by (4.2*b*)) for $-4 < \beta < -8/3$ (solid line) and M_∞ (defined by (4.22)) for $\beta < -4$ (dashed line) as a function of β in the limiting case $\lambda = \lambda_c$.

exist, but where λ is equal to the critical value λ_c at the source. These asymptotic critical similarity solutions are discussed in more detail below.

Since the limit of λ_l as $z \to \infty$ diverges from λ_s, if the value of λ is perturbed slightly from λ_c, then we deduce that the similarity solutions of §4.1 are unstable. In the next subsection we confirm this by analysing the stability of the solutions to small pertubations in the initial conditions.

4.4. *Stability of the similarity solutions*

To examine the stability of the similarity solutions defined by (4.1) to small perturbations in the initial conditions, it is convenient to scale the specific buoyancy flux F, the specific momentum flux M, and the volume flux Q with the similarity solutions (4.1), leading to the quantities

$$\bar{Q}(z) = \frac{Q(z)}{Q_u(z)}; \quad \bar{M}(z) = \frac{M(z)}{M_u(z)}; \quad \bar{F}(z) = \frac{F(z)}{F_u(z)}. \tag{4.9a–c}$$

Defining a new independent variable

$$\xi = \ln z, \tag{4.10}$$

the governing equations (2.14*a*–*c*) become autonomous and independent of λ_c:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\bar{Q} = -q(\bar{Q} - \bar{M}^{1/2}), \tag{4.11}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\bar{M} = -m\left(\bar{M} - \frac{\bar{F}\bar{Q}}{\bar{M}}\right), \tag{4.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\bar{F} = -f(\bar{F} - \bar{Q}). \tag{4.13}$$

The similarity solution corresponds to the unique non-trivial fixed point of the system defined by (4.11)–(4.13),

$$\bar{Q} = \bar{M} = \bar{F} = 1. \tag{4.14}$$

To analyse the stability of the similarity solution, we consider small perturbations about the fixed point defined as

$$(\bar{Q}, \bar{M}, \bar{F}) = (1, 1, 1) + \delta(Q', M', F'), \tag{4.15}$$

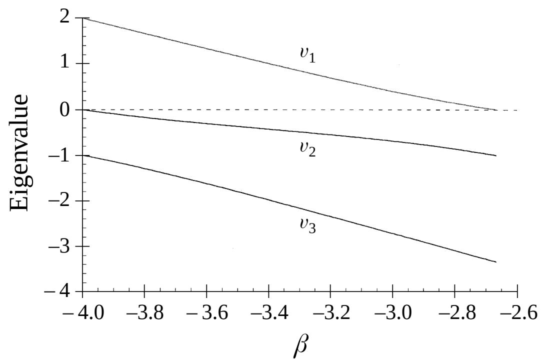

FIGURE 6. Variation of the eigenvalues of the stability matrix $\boldsymbol{A}$ (defined in (4.16)) with β.

where $\delta \ll 1$. Linearizing (4.11)–(4.13), we find that Q', M' and F' satisfy the relations

$$\begin{pmatrix} \dfrac{\mathrm{d}Q'}{\mathrm{d}\xi} \\ \dfrac{\mathrm{d}M'}{\mathrm{d}\xi} \\ \dfrac{\mathrm{d}F'}{\mathrm{d}\xi} \end{pmatrix} = \begin{pmatrix} -q & q/2 & 0 \\ m & -2m & m \\ f & 0 & -f \end{pmatrix} \begin{pmatrix} Q' \\ M' \\ F' \end{pmatrix} \equiv \boldsymbol{A} \begin{pmatrix} Q' \\ M' \\ F' \end{pmatrix}. \tag{4.16}$$

Assuming that the perturbations are proportional to $\exp(v\xi)$, then we find that the eigenvalues v_1, v_2, v_3, of the stability matrix $\boldsymbol{A}$ satisfy

$$v^3 + (2m + q + f)v^2 + \left(\frac{3mq}{2} + f[2m + q]\right) v + fmq = 0. \tag{4.17}$$

Figure 6 illustrates the variation of v_i with β for $-4 \leqslant \beta \leqslant -8/3$. Except in the limiting case $\beta = -8/3$, one eigenvalue (which we shall refer to as v_1) is positive, while the other two (v_2 and v_3) are negative.

Each of these eigenvalues has an associated eigenvector, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$, say in the space of volume, specific momentum and specific buoyancy fluxes $(\overline{Q}, \overline{M}, \overline{F})$. We deduce that, unless $\beta = -8/3$, any perturbation which includes a component of $\boldsymbol{e}_1$ is unstable, and the motion of the associated plume diverges from the similarity solution, as suggested by the numerical calculations in §3 and figure 4.

Furthermore, any small perturbation about $(1, 1, 1)$ on plane Π spanned by $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ is stable, and therefore, in physical space, the motion asymptotes to a similarity solution of the form defined by (4.1). If we define S to be the surface in $(\overline{Q}, \overline{M}, \overline{F})$-space which divides bounded and unbounded motion, then Π is tangent to this surface at $(1, 1, 1)$.

4.5. *The special case $\beta = -8/3$*

As noted above, in the limit $\beta = -8/3$, we recover the classical similarity solutions of Morton *et al.* (1956). Unlike the solutions with $\beta < -8/3$, in this special case $v_1 = 0$ and so the system has become marginally stable with eigenvalues

$$v_1 = 0, \qquad v_2 = -1, \qquad v_3 = -10/3. \tag{4.18a–c}$$

and associated eigenvectors

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \qquad \boldsymbol{e}_2 = \begin{pmatrix} 5/3 \\ 4/3 \\ 0 \end{pmatrix}, \qquad \boldsymbol{e}_3 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}. \tag{4.19a–c}$$

As a result of this marginal stability, for any initial perturbation in the direction $\boldsymbol{e}_1$, the system adjusts to a solution identical to the original solution, except that there is a simple rescaling of the volume flux and the specific momentum flux in terms of the perturbation to the specific buoyancy flux. The eigenvector $\boldsymbol{e}_2$ now has associated eigenvalue $v_2 = -1$, and so perturbations of this form decay with distance like $\exp(-\xi) = 1/z$. Therefore, for such a perturbation, in physical space, the volume flux and specific momentum flux have the form

$$Q \sim z^{5/3}\left(1 + \frac{5\delta}{3z}\right), \tag{4.20}$$

$$M \sim z^{4/3}\left(1 + \frac{4\delta}{3z}\right). \tag{4.21}$$

This is exactly the linearized form of the solution for a plume with finite initial volume flux and/or finite initial specific momentum flux rising from a real source and asymptoting to a 'pure' plume which appears to be rising from a point source at height $-\delta$, the so-called 'effective origin' (see Caulfield & Woods 1995 for a fuller discussion). Perturbations along vector $\boldsymbol{e}_3$ correspond to a perturbation with infinite negative initial momentum flux which is unphysical.

This analysis provides some insight into why the similarity solution for a buoyant plume rising in an unstratified environment is marginally stable, and may therefore arise in nature or the laboratory. In contrast, it also suggests that in an environment of decreasing stratification, $\beta < -8/3$, the similarity solutions are unstable, and will not be seen in the laboratory: instead the motion is either bounded, or tends to the solution for a plume rising through an unstratified environment.

4.6. *Asymptotic similarity solutions when $\lambda = \lambda_c$ for $\beta < -4$*

The similarity solutions (4.1), which only exist for $-4 < \beta < -8/3$, may be identified with the boundary between bounded and unbounded plumes. However, if $\lambda > 0$ then when $\beta = -4$, these similarity solutions (4.1) become degenerate and are only approached at an infinite distance from the source of (finite) initial specific buoyancy flux, where they are predicted to have constant but unbounded specific momentum flux. Thus we expect a different asymptotic regime to describe the critical plume solutions with $\lambda = \lambda_c$ and $\beta < -4$.

When $\beta < -4$ the numerical solutions with $\lambda = \lambda_c$ are unbounded with the specific buoyancy flux $F \to 0$ as $z \to \infty$. Therefore, we expect that they also asymptote to a self-similar form. Although equations (2.14a–d) have no exact self-similar solutions for $\beta < -4$, it may be shown that, for each λ_c, there is a family of asymptotic similarity solutions given by

$$M \simeq M_\infty(\lambda_c), \tag{4.22}$$

$$Q_\infty(z) \equiv Q_a(z) \simeq M_\infty^{1/2} z, \tag{4.23}$$

$$F_\infty(z) \equiv F_a(z) \simeq \frac{\lambda_c^\beta M_\infty^{1/2} z^{\beta+2}}{|\beta + 2|}, \tag{4.24}$$

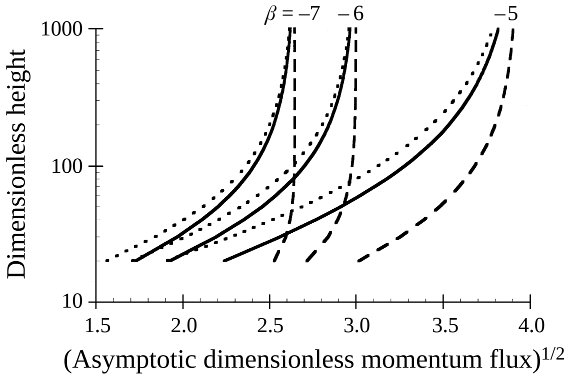

FIGURE 7. Comparison of $|\beta+2|\lambda_c^{-\beta}Fz^{|\beta+2|}$ (solid line), $M^{1/2}$ (dashed line) and Q/z (dotted line) as a function of z for three different values of β in the case $\lambda = \lambda_c$ with $\beta < -4$ (§4.6). Note that in each case the three quantities are asymptoting to a constant, indicative of the validity of the asymptotic solution (4.22)–(4.24).

where M_∞ is constant. These solutions apply provided z is sufficiently large

$$z \gg \left(\frac{M_\infty|\beta+2||\beta+4|}{\lambda_c^{|\beta|}}\right)^{1/|\beta+4|}. \qquad (4.25)$$

We have verified numerically that, in the case $\lambda = \lambda_c$, the numerical solutions of §3 asymptote to the self-similar form (4.22)–(4.24) by determining that, as $z \to \infty$,

$$M^{1/2}, \quad Q/z, \quad |\beta+2|\lambda_c^{-\beta}Fz^{|\beta+2|} \to M_\infty^{1/2}. \qquad (4.26)$$

We show this convergence for three different values of β in figure 7. The results shown in figure 2 also confirm that for $\lambda < \lambda_c$ the numerical solutions are bounded and hence do not converge to these asymptotic similarity solutions, while for $\lambda > \lambda_c$ the numerical solutions are unbounded and the specific buoyancy flux converges to a finite positive value as $z \to \infty$ thereby again diverging from the asymptotic similarity solutions. In figure 5 we also show (dashed line) the variation of M_∞ as a function of β in the case $\lambda = \lambda_c(\beta)$. It is seen that M_∞ decays monotonically to zero as β decreases to values smaller than -4.

Also note that for this solution, $\lambda_l(z) \to \infty$ as $z \to \infty$ (see (4.6)), consistent with the result $\lambda_s \to \infty$ as $\beta \to -4$ (4.7). Also, as $\beta \to -4$ from below, (4.25) implies that the solution is only formally valid at infinity, which is consistent with the limit for (4.8) as $\beta \to -4$ from above.

Physically these asymptotic similarity solutions correspond to plumes which have just sufficient specific buoyancy flux to propagate through the strongly stratified region near the source, while continuing to gain specific momentum flux and volume flux. Then, as they continue to propagate far from the source, where the ambient fluid is very weakly stratified, the specific momentum flux becomes essentially constant, as there is very little density difference between the ambient and plume fluid, while the volume flux continues to increase through entrainment.

As can be seen from figure 3, as β decreases below -4, λ_c also decreases, and is always significantly less than unity. This implies that the stratification decreases more slowly over the height to which the plume would rise if the environment were of uniform stratification. Therefore, the loss of specific buoyancy flux occurs nearer the source as β decreases for these critical solutions with $\lambda = \lambda_c$. Above this region, the specific momentum flux of the plume is expected to remain approximately constant and so the value of M_∞ decreases with β (figure 5).

As with the similarity solutions (4.1), the asymptotic solutions defined by (4.22)–(4.24) are also unstable to perturbations in the initial conditions; indeed, we have seen that for $\lambda < \lambda_c$ the motion is bounded, whereas for $\lambda > \lambda_c$, the plume asymptotes to the motion of a plume of fixed specific buoyancy flux rising through an unstratified environment.

5. Thermal motion

When there is an instantaneous and finite release of buoyancy, a discrete thermal cloud develops above the source. In their seminal contribution, Morton *et al.* (1956) also developed a model of thermals rising in both uniform and uniformly stratified environments. Following the approach of §§2–4, we now extend this model to describe the ascent of thermals through a non-uniformly stratified environment.

The motion of a thermal may be described by the (dimensional) equations

$$\frac{\mathrm{d}V}{dz} = 3\epsilon_t V^{2/3}, \quad \frac{\mathrm{d}P}{dz} = \frac{2}{3}\frac{BV}{P}, \quad \frac{\mathrm{d}B}{dz} = -VN^2, \tag{5.1a–c}$$

where $\epsilon_t \sim 0.25$ is the thermal entrainment constant (Morton *et al.* 1956), and the volume V, specific momentum P, and specific buoyancy B of the thermal are defined in terms of the characteristic velocity w, the effective radius b and the reduced gravity g' according to (Morton *et al.* 1956)

$$V \equiv b^3, \quad P \equiv b^3 w, \quad B \equiv b^3 g'. \tag{5.2a–c}$$

Following the earlier parts of the paper, we now consider the behaviour of thermals rising through an ambient with stratification of the form

$$N^2 = N_s^2 (z/z_s)^\beta. \tag{5.3}$$

In the special case $N^2 = N_s^2$, a constant, the maximum height of rise of a thermal with initial conditions

$$V = P = 0, \quad B = B_s \quad \text{at} \quad z = 0 \tag{5.4}$$

is (Morton *et al.* 1956)

$$H_{mt} = 6^{3/4} H_t, \tag{5.5}$$

where H_t is the characteristic height of rise in a uniformly stratified environment

$$H_t \equiv (3\epsilon_t)^{-3/4} B_s^{1/4} N_s^{-1/2}. \tag{5.6}$$

H_t has been chosen for convenience to define non-dimensional variables

$$\hat{B} = \frac{B}{B_s}, \quad \hat{V} = \frac{V}{(3\epsilon_t H_t)^3}, \quad \hat{P} = \frac{P}{(18\epsilon_t^3 B_s)^{1/2} H_t^2}, \quad \hat{z} = \frac{z}{H_t}, \quad \hat{N} = \frac{N}{N_s}, \tag{5.7a–e}$$

which yield the simple non-dimensional form for the governing equations

$$\frac{\mathrm{d}}{\mathrm{d}\hat{z}}\hat{V} = \hat{V}^{2/3}, \quad \hat{P}\frac{\mathrm{d}}{\mathrm{d}\hat{z}}\hat{P} = \hat{B}\hat{V}, \quad \frac{\mathrm{d}}{\mathrm{d}\hat{z}}\hat{B} = -\hat{V}\hat{N}^2, \quad \hat{N}^2 = (\mu\hat{z})^\beta. \tag{5.8a–d}$$

In the remainder of this section of the paper, we work with non-dimensional quantities, and therefore for convenience, again we drop the hat notation.

The parameter μ is the direct analogue of the parameter λ for plumes (2.15), and represents the ratio of the height of rise of a thermal in a uniformly stratified environment to the length scale of variation of the buoyancy frequency

$$\mu = \frac{H_t}{z_s}. \tag{5.9}$$

5.1. *Exact solutions for a point-source thermal*

As in §3, to compare our results with the classical results of Morton *et al.* (1956), we define the thermal source to be at the point $z = 1/\mu$, and we focus on the initial conditions

$$B = 1, \quad V = 0, \quad P = 0. \tag{5.10}$$

Given these boundary conditions, (5.8*a–d*) have a straightforward, but algebraically complex, analytical solution. This analytical solution enables us to derive exactly analogous results for thermal motion to the numerically and asymptotically calculated results for the buoyant plumes discussed in the previous sections.

Equation (5.8*a*) can be integrated, using (5.10), to obtain

$$V = \left(\frac{z}{3} - \frac{1}{3\mu}\right)^3. \tag{5.11}$$

Substituting (5.11) into (5.8*c*) (using (5.8*d*)) enables us to get a polynomial solution for B, which when substituted into (5.8*b*) along with (5.11) allows us to obtain a closed form solution for P^2. Provided $\beta \neq -1, -2, -3, \ldots -8$, the exact solution takes the form

$$P = \left(P_c + \frac{B_c}{54}\left(z - \frac{1}{\mu}\right)^4 + P_t\right)^{1/2}, \tag{5.12}$$

$$B = B_c + B_t, \tag{5.13}$$

where

$$P_c = \frac{280}{81\mu^8(\beta+1)(\beta+2)(\beta+3)(\beta+4)(\beta+5)(\beta+6)(\beta+7)(\beta+8)}, \tag{5.14}$$

$$\begin{aligned} P_t = -\frac{2z^{\beta+2}\mu^\beta}{729}\Bigg[&\frac{z^6}{(\beta+4)(\beta+8)} - \frac{3(2\beta+7)z^5}{\mu(\beta+3)(\beta+4)(\beta+7)} \\ &+\frac{3(5\beta^2+30\beta+42)z^4}{\mu^2(\beta+2)(\beta+3)(\beta+4)(\beta+6)} \\ &-\frac{2(5\beta^2+25\beta+21)(2\beta+5)z^3}{\mu^3(\beta+1)(\beta+2)(\beta+3)(\beta+4)(\beta+5)}, \\ &+\frac{3(5\beta^2+20\beta+17)z^2}{\mu^4(\beta+1)(\beta+2)(\beta+3)(\beta+4)} \\ &-\frac{3(2\beta+3)z}{\mu^5(\beta+1)(\beta+2)(\beta+3)} + \frac{1}{\mu^6(\beta+1)(\beta+2)}\Bigg], \end{aligned} \tag{5.15}$$

$$B_c = 1 - \frac{2}{9\mu^4(\beta+1)(\beta+2)(\beta+3)(\beta+4)}, \tag{5.16}$$

$$B_t = -\frac{\mu^\beta}{27}\left[\frac{z^{\beta+4}}{(\beta+4)} - \frac{3z^{\beta+3}}{\mu(\beta+3)} + \frac{3z^{\beta+2}}{\mu^2(\beta+2)} - \frac{z^{\beta+1}}{\mu^3(\beta+1)}\right]. \tag{5.17}$$

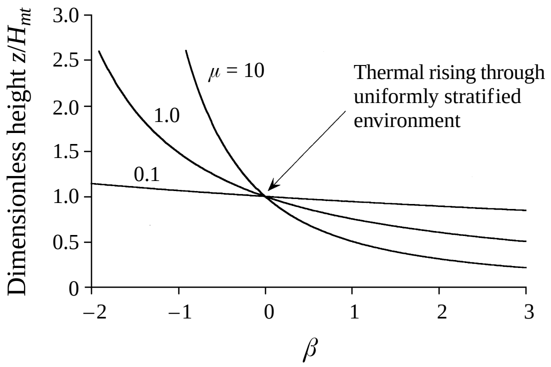

FIGURE 8. Non-dimensional height of rise of a turbulent buoyant thermal in an environment of non-uniform stratification (defined by (1.1)) as a function of β. Curves are given for three values of μ (defined by equation (5.9)), namely 0.1, 1.0 and 10.0. For reference, the plume heights have been scaled relative to the height of rise H_{mt} (defined by (5.5)) of a thermal rising in a uniformly stratified environment (cf. Morton *et al.* 1956).

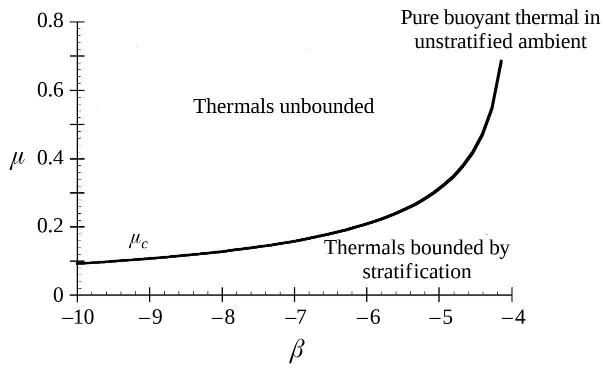

FIGURE 9. Variation of the critical value μ_c (defined by (5.18)) separating bounded and unbounded solutions as a function of β.

In figure 8, we illustrate how the height of rise of the thermal, h_t, defined as the point at which $P(h_t) = 0$, varies with β. As expected, the qualitative characteristics of this plot are very similar to those for turbulent plumes (figure 1).

If $\beta > -4$, then both B_t and P_t are negative, and increase in amplitude without limit as z increases, and so the thermal will eventually become dense and stop at some finite height. On the other hand, for rapidly decaying ambient stratifications, $\beta < -4$, (5.17) implies that B_t is positive, and decays towards zero as $z \to \infty$, and hence if $B_c > 0$ the thermal is unbounded. The condition that $B_c > 0$ implies that for μ greater than a critical value μ_c, defined as

$$\mu_c(\beta) \equiv \left(\frac{2}{9(\beta+1)(\beta+2)(\beta+3)(\beta+4)}\right)^{1/4}, \tag{5.18}$$

then the thermal is unbounded. The value of μ_c is plotted in figure 9 as a function of β.

The analytical solution given above enables us to determine directly the asymptotic behaviour of the unbounded thermals in both the cases $\mu > \mu_c$ and $\mu = \mu_c$, as well as some properties of the bounded thermals, $\mu < \mu_c$. For $\beta < -4$, $P_c + P_t$ increases as z increases for all z. However, since $P_t = O(z^{\beta+8})$, then for sufficiently large z the specific momentum is totally dominated by the term involving B_c, which is $O(z^4)$.

(i) $\mu < \mu_c$. In this case $B_c < 0$, and so the motion is bounded, and the thermal is only able to ascend to a finite height, $h_t(\mu)$. Direct calculations identify that as $\mu \to \mu_c$ from below, $h_t(\mu) \to \infty$, again in a directly analogous fashion to that for a plume (figure 2*b*).

(ii) $\mu > \mu_c$. If the motion is unbounded, then the solution (5.11)–(5.17) identifies that, as $z \to \infty$, the motion asymptotes to that of a self-similar thermal in an unstratified environment (cf. Morton *et al.* 1956), with total specific buoyancy B_c, specific momentum

$$P_a \simeq \frac{B_c^{1/2} z^2}{3\sqrt{6}}, \tag{5.19}$$

and volume

$$V_a \simeq \frac{z^3}{27}. \tag{5.20}$$

The expression (5.16) identifies that as $\mu \to \mu_c$ from above, $B_c \to 0$; as expected, this result is directly analogous to that for a plume (figure 2*a*).

(iii) $\mu = \mu_c$. With the exact solution (5.11)–(5.17), we can deduce the asymptotic behaviour of the critical solutions with $\mu = \mu_c$ directly. Asymptotically the volume has the value given by (5.20) and the specific buoyancy for large z approaches

$$B_a \simeq -\frac{\mu_c^\beta z^{\beta+4}}{27(\beta+4)}. \tag{5.21}$$

In this case, we need to distinguish between two different circumstances, depending on the asymptotic properties of the term P_t defined by (5.15).

If $-8 < \beta < -4$, P_t is positive for large z, and grows in magnitude with increasing z so that asymptotically

$$P_a \simeq \frac{\sqrt{2}\mu_c^{\beta/2} z^{(\beta+8)/2}}{27[|\beta+4|(\beta+8)]}. \tag{5.22}$$

Alternatively, when $\beta < -8$, then P_t is negative, but decays towards zero with increasing z, and so P asymptotes to the constant value (using (5.12), (5.14) and (5.18))

$$P_a \simeq \left(\frac{70(\beta+1)(\beta+2)(\beta+3)(\beta+4)}{(\beta+5)(\beta+6)(\beta+7)(\beta+8)}\right)^{1/2}, \tag{5.23}$$

Thus the thermal approaches the behaviour of a pure momentum thermal in an unstratified environment. As for the plume (§4.6), this asymptotic momentum $P_a \to \infty$ as $\beta \to -8$, and decreases monotonically as β decreases for $\beta < -8$.

5.2. *Similarity solutions for thermals*

As with turbulent plumes, the boundary between bounded and unbounded motion $\mu = \mu_c$ is associated with a family of (unstable) similarity solutions which coincide with the asymptotic form of the critical full solution, given by (5.20)–(5.22). However, for β in the range $-8 < \beta < -4$, these similarity solutions exist for general μ and may be found by direct substitution into the governing equations (5.8*a*–*d*). The solutions are given by

$$V_u = \frac{z^3}{27}, \quad P_u = \left(\frac{2}{3}\right)^{1/2} \frac{\mu^{\beta/2}\mu_s^2 z^{(\beta+8)/2}}{3(\beta+8)^{1/2}}, \quad B_u = \mu^\beta \mu_s^4 z^{\beta+4}, \tag{5.24a–c}$$

where

$$\mu_s = \left(\frac{1}{27|\beta+4|}\right)^{1/4}. \tag{5.25}$$

In the limiting case, $\beta = -4$, these similarity solutions become unbounded except in the special case $\mu(\beta+4)^{1/2} = 0(1)$ when they coincide with the solutions for a buoyant thermal rising in an unstratified environment (cf. Morton *et al.* 1956), given by (5.19) and (5.20), with $B = B_c$ a constant. In the other limit, $\beta \to -8$, the solution again diverges, except for the special case $\mu(\beta+8)^{1/8} = O(1)$, when the similarity solution corresponds to a forced neutrally buoyant instantaneous release of fluid, i.e. $P = P_c$ a constant and $B = 0$.

By following a similar procedure to that described in §4, it may be shown that small perturbations to the similarity solutions (5.24) for a buoyant thermal are unstable, except in the limiting case $\beta = -4$, for which the similarity solution is neutrally stable. Essentially, a general small perturbation away from the exact initial conditions for the similarity solution leads to either bounded motion or asymptotes towards the unbounded motion of a thermal of constant specific buoyancy in a uniform environment. Furthermore, although the solution (5.11)–(5.17) asymptotes to the self-similar solution (5.24) as $z \to \infty$ when $\mu = \mu_c$, if $\mu \neq \mu_c$, then the two solutions diverge. This may be seen by considering the evolution with height of the local thermal property

$$\mu_l(z) = \frac{B^{1/4}N(z)^{-1/2}}{z} \tag{5.26}$$

(cf. the quantity $\lambda_l(z)$, defined by (4.6)). From the solution (5.11)–(5.17), we see that if $\mu < \mu_c$ then $\mu_l(z)$ decreases, and becomes zero when the thermal reaches the neutral buoyancy height. On the other hand, if $\mu > \mu_c$ then, since $B \to B_c > 0$ (as defined by (5.16)),

$$\mu_l \simeq B_c^{1/4}\mu^{-\beta/4}z^{-(\beta+4)/4} \to \infty, \tag{5.27}$$

provided $\beta < -4$. However, in the special case $\mu = \mu_c$, $B \simeq B_a$ as $z \to \infty$, as defined in equation (5.21). Therefore, $\mu_l \to \mu_s$ as $z \to \infty$ and the solution asymptotes to a self-similar solution of the form given by (5.24) for $-8 < \beta < -4$.

Finally, note that for $\beta < -8$, the solution (5.11)–(5.17) with $\mu = \mu_c$ asymptotes to the asymptotic similarity solution in which P is given by the constant value (5.23), and B is given by (5.21).

6. Applications and discussion

Recognition of the critical value λ_c dividing bounded and unbounded plumes when $\beta \leqslant -8/3$ is of particular importance in understanding the fate of real plumes in environments of non-uniform stratification. In order to apply the results to real problems, it is useful to return to dimensional variables, and this leads to the condition that, for a plume to continue propagating through a region of decaying stratification,

$$F_s > 2^{5/2}\epsilon_p^2 N_s^3 \left(\lambda_c(\beta)z_s\right)^4. \tag{6.1}$$

This relation has the simple physical interpretation that when the vertical scale over which the stratification decays is smaller than the scale over which the plume would rise in an environment with stratification equal to that at the source, the plume is able to continue rising indefinitely, even though the environment is stably stratified at all heights. Also note that for very slowly decaying stratification, when $\beta > -8/3$, the plume is always bounded.

In §1, we mentioned a number of situations in which the ambient stratification is non-uniform. In such situations, the height of rise of a plume may be very different from that estimated using the solutions of Morton *et al.* (1956) for a uniform environment, as can be seen in figure 1. In the oceanic context, an important application concerns the downward propagation of turbidity currents on continental shelves, plumes of dense brine rejected as sea-ice forms in polar regions (Morison *et al.* 1992) and plumes of dense waste material deposited into the ocean. As an example, we may approximate the variation of the buoyancy frequency with depth in the upper 500 m of the subtropical Northern Summer Pacific as presented by Emery *et al.* (1984) with the power law,

$$N^2 = N_s^2(z/z_s)^{-2.75}, \tag{6.2}$$

with $N_s^2 = 0.0004$ s^{-2} and $z_s \sim 100$ m. Substituting these values into expression (6.1), and reading an appropriate value of λ_c from figure 3, we deduce that a sinking plume would require a specific buoyancy flux in excess of about $F_s \sim 200$ m^4 s^{-3} in order to penetrate through the thermocline into the less-stratified deep ocean. For a typical reduced gravity g' of 0.1 m s^{-2}, this requires an actual volume flux of 6000 m^{3}s^{-1}. If material is injected more slowly, then the specific buoyancy flux will be insufficient for the plume to penetrate through the thermocline into the deeper ocean.

The density profile in ventilated rooms often develops non-uniform stratification as a result of combinations of heat sources and sinks at different positions in the room (Lane-Serff 1989; Linden & Cooper 1996; Baines & Turner 1969). Figure 1 illustrates that the height of penetration of a plume can vary by over an order of magnitude if the stratification in the room is non-uniform. As a result of this variation, in naturally ventilated buildings, the depth of mixed layers and the dispersal of smoke can vary enormously depending on the strength and location of heat sources. Typical values of vertical temperature variation within a room are of the order of 1–3 K m^{-1} (Lane-Serff 1989), with the larger values occurring at levels within a room close to the height of openings to external air of different temperature. Fitting a power-law distribution to a particular but typical laboratory-scale analogue experiment (Lane-Serff 1989) the stratification can be approximated by a power-law distribution of the form

$$N^2 = N_s^2(z/z_s)^{-1.7}. \tag{6.3}$$

When the experiments are scaled appropriately to be applicable to the density distribution within a real room, $N_s^2 = 0.01$ s^{-2} and $z_s \sim 1$ m.

The present model identifies that in a room in which the stratification decays sufficiently rapidly with height, the plume rises indefinitely and asymptotes to the motion of a simple plume in a uniform environment. For the above-quoted directly measured stratifications, even very weak sources of buoyancy (with initial volume fluxes of the order of 10 cm^{3}s^{-1}, and temperature anomalies of the order of 10 K) are sufficiently 'strong' to penetrate to the top of a room of depth 5 m. Therefore, sufficiently far above the source, the volume flux in the plume is given by the asymptotic relation $Q(z) \sim z^{5/3}F_0^{1/3}$, where F_0 is the initial buoyancy flux. Importantly, even for a typical non-uniformly stratified closed room, the development of such a plume drives a filling-box-type return flow with the descent speed of the first front being well-predicted directly using the results of Baines & Turner (1969). Finally, we note that the effectiveness of industrial chimney plumes in driving pollutants high into the atmosphere is strongly controlled by the stratification. As inversions develop, and the stratification in the lower few kilometres of the atmosphere increases, only very large plumes are able to continue ascending.

7. Conclusions

We have examined the role of ambient stratification on the ascent of turbulent buoyant plumes. We have established is that if the environmental stratification decays sufficiently rapidly above a source of buoyancy, then, even though the environment is stratified, the plume is able to ascend indefinitely. However, if the ambient stratification decays sufficiently slowly relative to the scale of the plume, then the motion of the plume is confined within a finite distance of the source, as occurs in a uniformly stratified environment. We have also identified a new class of similarity solutions describing the ascent of plumes in a stratified environment. These solutions are associated with the boundary between bounded and unbounded plumes. The solutions are unstable in the sense that small perturbations in the initial conditions cause the motion to diverge from the self-similar solution. We have applied the results to determine critical conditions for plumes to penetrate the thermocline in the upper ocean, and discussed the relevance of the results in a wider context.

Richard Kerswell gave some useful suggestions about the stability results presented herein. Our research is supported by the NERC, EPSRC, the Leverhulme and Nuffield Foundations and the Royal Society. We would like to express our gratitude to one of the referees for bringing the pioneering paper of Zeldovich (1937) to our attention.

Appendix. Atmospheres with exponential density gradients

In the main text we examined the ascent of buoyant plumes and thermals through atmospheres in which the stratification varies algebraically with height. For completeness, here we describe the ascent of plumes and thermals through an environment in which the stratification varies exponentially with height. We consider

$$N^2 = N_s^2 \exp(-z/z_s). \qquad \text{(A 1)}$$

We adopt the same non-dimensional variables as in §§2 and 5, so that for plumes the specific buoyancy flux F, the specific momentum flux M and the volume flux Q evolve with height according to

$$\frac{\mathrm{d}Q}{\mathrm{d}z} = M^{1/2}, \quad \frac{\mathrm{d}M}{\mathrm{d}z} = \frac{QF}{M}, \quad \frac{\mathrm{d}F}{\mathrm{d}z} = -Q\mathrm{e}^{-\lambda z}, \qquad \text{(A 2}a\text{–}c\text{)}$$

with λ as defined in (2.15). As expected, λ is still the key parameter in this model. For small values of λ, the plume experiences a small decrease in the ambient stratification with height and so the plume height only increases a small amount relative to the case of a uniform environment. However, with larger values of λ, the plume height increases much more significantly owing to the progressively less stratified environment which the plume encounters. Eventually, for $\lambda > \lambda_e$, the ambient stratification decays so rapidly relative to the evolution of the plume, that the plume is able to continue rising and becomes unbounded. Numerical calculations show that $\lambda_e = 0.847$.

For thermals, we are able to derive an analogous result analytically, in a similar fashion to §5. With initial conditions $B = 1$, $V = 0$, we only need consider the equation for the specific buoyancy, which takes the form (cf. §5)

$$\frac{\mathrm{d}}{\mathrm{d}z}B = -\frac{1}{27}z^3 \mathrm{e}^{-\mu z}. \tag{A 3}$$

This can be integrated to yield

$$B = 1 + \frac{2}{9\mu^4}(-1 + \mathrm{e}^{-\mu z}) + \frac{z\mathrm{e}^{-\mu z}}{27}\left(\frac{z^2}{\mu} + \frac{3z}{\mu^2} + \frac{6}{\mu^3}\right). \tag{A 4}$$

so that the thermal is able to escape the effects of the stratification for $\mu > \mu_e$, where

$$\mu_e = \left(\frac{2}{9}\right)^{1/4} = 0.6866. \tag{A 5}$$

We deduce that in an exponentially stratified environment, a plume with sufficient specific buoyancy flux,

$$F_s \geqslant (\lambda_e z_s)^4 N_s^{-3}, \tag{A 6}$$

can propagate indefinitely, while a thermal requires initial specific buoyancy

$$B_s \geqslant (\mu_e z_s)^4 N_s^{-2} \tag{A 7}$$

to continue to propagate.

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